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Calculus of Variations and Optimal Control Theory*)

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When John Bernoulli amused himself in 1696 with the design of a slide of such shape as to permit the proverbial, infinitely smooth, elephant of neglectible volume to slide from point A to point B (Figure 1) in the shortest possible time, a new mathematical discipline, the <u>Calculus of Variations</u>, came into existence.

In the middle of the 18th century, this problem was subjected to generalizations and analysis by L. Euler and J. L. de Lagrange. An example of such a more general problem is the so-called problem of Lagrange, namely, to find a vector function $y: R \to R^n$ (curve in $(n + 1)$-dimensional (t,y)-space) which satisfies specified boundary conditions at $t = 0$ and for some $t = T > 0$, satisfies a set of constraining differential equations of first order,

(1) $$\phi(t,y,y') = 0, \text{ where } \phi: R \times R^n \times R^n \to R^\mu, \quad \mu < n,$$

and is such that

(2) $$I[y] = \int_0^T f(t,y(t),y'(t))dt, \quad f: R \times R^n \times R^n \to R,$$

yields the smallest value that is attainable. We shall hereby assume without much loss of generality, that f, ϕ and their partial derivatives of first order with respect to all variables are continuous.

For this problem to have meaning, one has to specify the type of functions that are admitted to compete for the minimum of (2). In the classical theory, one usually required that y' be sectionally continuous ($y' \in S$). Then, the solution y, if it exists, is sectionally smooth. (The understanding is that $y(t) = y(0) + \int_0^t y'(s)ds$.) [1, p. 187ff, 12, p. 325ff.] On the other hand, one could permit y' to be bounded and measurable ($y' \in M$). Then, y is absolutely continuous [5]. An even more general case is the one where the so-called generalized curves which were introduced by L. C. Young are permitted to compete [8].

A number of necessary and sufficient conditions for a function to yield a minimum for (2) have been derived over the past 250 years. The more recent work consisted chiefly in generalizing, refining, and "rigor-proofing" the work of the 18th and early 19th century mathematicians.

The most prominent and most widely known necessary conditions are the multiplier rule and the Weierstrass condition.

The multiplier rule states that if y is a solution with $y' \in S$, then there exists a vector function $\lambda: R \to R^{\mu}$, $\lambda \in S$ and a constant λ_0, where $(\lambda_0, \lambda) \neq (0,0)$, so that the equations

$$(3) \quad \lambda_0 f_{y'}(t,y(t),y'(t)) + \lambda.\phi_{y'}(t,y(t),y'(t)) = \\ = \int_0^t [\lambda_0 f_y(t,y(t),y'(t)) + \lambda.\phi_y(t,y(t),y'(t))]dt + C$$

are satisfied wherever y' is continuous and where C is some constant n-vector [12, p. 334]. (If $y' \in M$, then $\lambda \in M$ and the equations (3) have to hold almost everywhere on [0,T] [8, p. 24]. In either case, λ is continuous wherever y' is continuous. (Equations (3) in differentiated form are often referred to as the Euler-Lagrange equations or the Mayer equations.)

The Weierstrass condition states that, with $\lambda_0 \geq 0$, whenever y' is continuous,

$$(4) \quad \lambda_0[f(t,y(t),\zeta) - f(t,y(t),y'(t))] + (y'(t) - \xi)\cdot[\lambda_0 f_{y'}(t,y(t),y'(t)) + \\ + \lambda.\phi_{y'}(t,y(t),y'(t))] \geq 0$$

has to hold for all ξ for which $\phi(t,y(t),\xi) = 0$. If $y' \in M$, then (4) has to hold a.e. on [0,T] [8, p. 24].)

If the beginning point of y at t = 0 and the endpoint for some t = T are allowed to move freely on some specified manifolds, some additional conditions, the so-called <u>transversality conditions</u> [3, p. 24] have to be satisfied at $t = 0$ and $t = T$. These conditions take the place of the (non-existent) boundary conditions.

Let it be noted that in the formulation as well as in the analysis of this problem that leads to (3) and (4), no restrictions other than (1) are placed upon the values of $y'(t)$. This is essentially equivalent with assuming that the values of $y'(t)$ lie in an open subset of R^n.

The Lagrange problem (1), (2) and the two necessary conditions (3) and (4) may be interpreted as a (vast) generalization of the problem of finding the minimum of a (differentiable) real-valued function of a real variable in an <u>open</u> interval.

Let $h: (-1,1) \to R$, h differentiable, assume its minimum for $t = t_0$. Then, by necessity,

$$h'(t_0) = 0 \tag{3a}$$

$$h(t_0) \leq h(\xi) \text{ for all } \xi \in (-1,1). \tag{4a}$$

This analogy is not nearly as far-fetched as it would appear upon first glance. Actually, the left side of (3) is obtained by a process that is, in essence, a generalization of the differentiation process to functionals that are defined on a subset of the solution set of (1), and (4) is, in effect, a necessary condition for

$$\int_0^T f(t,\bar{y}(t),\bar{y}'(t))dt - \int_0^T f(t,y(t),y'(t))dt \geq 0$$

to hold for all solutions of $\phi(t,\bar{y}(t),\bar{y}'(t)) = 0$ that pass for $t = 0$ through the same beginning point and for $t = T$ through the same endpoint as y.

Pure mathematical speculation as well as practical problems that were generated by a modern and very sophisticated technology led to a re-examination of the scope and the applicability of the Calculus of Variations. To illustrate this, let us first mention that a continuous function need not assume its minimum in an open interval, but does assume its minimum in a closed and bounded interval. Similarly, there are problems which may be formulated as Variational Problems if the admissible values of $y'(t)$ form an open set, that have no solution, but do have a solution if the values of $y'(t)$ are restricted to a closed and bounded set. This is dramatically demonstrated by the example of applying a suitable external force, say x_3', to a moving masspoint, whose location and velocity are given by $(y(t),y'(t))$, to make it move from a given initial state $y(0) = x_1^0$, $y'(0) = x_2^0$ to the terminal state $(0,0)$ obeying the law of motion $y'' = x_3'(t)$, in the shortest possible time. With $y = x_1$, $y' = x_2$, we may formulate this problem as follows: To find (x_1, x_2, x_3) so that $x_1(0) = x_1^0$, $x_2(0) = x_2^0$, $x_1(T) = 0$, $x_2(T) = 0$ for some $T > 0$,

$$(5) \qquad \begin{aligned} x_1' &= x_2 \\ x_2' &= x_3' \end{aligned}$$

such that

$$(6) \qquad \int_0^T dt$$

is as small as possible.

This problem which is of the type (1), (2), does <u>not</u> have a solution if the admissible values of $x_3'(t)$ form an open set, as the reader can easily convince himself. However, it does have a solution when the values of $x_3'(t)$ are - more realistically - restricted to a compact set, say by $|x_3'(t)| \leq 1$, as was first shown by D. Bushaw [2]. But then, this problem ceases to be a variational problem and it is not immediately amenable to the analysis that leads to (3) and (4), as well as to other necessary and sufficient conditions in the Calculus of Variations.

This latter problem is characteristic of the type of problems that are dealt with in what is now referred to as the <u>Modern Theory of Optimal Control</u>, a new mathematical discipline that exploded into being in the 1950's. Generally, the theory of optimal control deals with problems such as the following one: To find a vector function (control) $u: R \to R^m$ with $u(t) \in U$ where U (control region) is a given subset of R^m so that the solution $x: R \to R^n$ (trajectory) of

$$(7) \qquad x' = \Phi(t,x,u(t)), \quad \Phi: R \times R^n \times R^n \to R^n,$$

satisfies certain boundary conditions at $t = 0$ and for some $t = T > 0$ and is such that

$$J[u] = \int_0^T F(t,x(t),u(t))dt, \quad F: R \times R^n \times R^m \to R, \tag{8}$$

yields the smallest attainable value. Again, we shall assume that F, Φ and their partial derivatives of first order with respect to all variables are continuous. For competition, we either admit $u \in S$, or $u \in M$. In the first case, the trajectory - if it exists - is sectionally smooth, in the latter case it is absolutely continuous.

Were it not for the restriction $u(t) \in U$, or if U were open, this problem would simply be a special case of the Lagrange problem (1), (2). This can be easily seen by setting $u = z'$, $x' - \Phi(t,x,u) = \phi(t,x,x',z')$, $F(t,x,u) = f(t,x,z')$.

However, if U is not open or if it is not known apriori that $u(t)$ lies in the interior of U for all t, then the classical arguments that were developed for the analysis of variational problems are no longer applicable and one faces, what appears to be an entirely new type of problem, superficial similarities notwithstanding.

Although successful attempts have been made to deal with special cases of such problems within the framework of the Calculus of Variations many years before the official birth of optimal control theory (see for example [13] in conjunction with [7]), a unified, comprehensive, and persuasively presented treatment had not appeared (in English) prior to 1962. At that time, L. S. Pontryagin and his collaborators published a theory, divorced from the classical Calculus of Variations, the most widely known aspect of which is what has become known as <u>Pontryagin's Principle</u> [11].

Pontryagin's principle states that for $u \in S$, $u(t) \in U$ to be a solution of the optimal control problem (7), (8) (<u>optimal control</u>), it is necessary that there exists a sectionally smooth vector function $p: R \to R^n$ and a constant $p_0 \geq 0$ with $(p_0, p) \neq (0,0)$ so that

$$(9) \qquad p' = -p_0 F_x(t,x(t),u(t)) - p \cdot \Phi_x(t,x(t),u(t))$$

for all t where u is continuous and

$$(10) \quad p_0 F(t,x(t),u(t)) + p \cdot \Phi(t,x(t),u(t)) \leq p_0 F(t,x(t),\xi) + p \cdot \Phi(t,x(t),\xi)$$

for all $\xi \in U$ and all t for which u is continuous. (If $u \in M$, then p is absolutely continuous and (9), (10) have to hold a.e. on $[0,T]$.) [11, p. 61, 81].

Again, if U is open, or the entire R^m space, then (9) and (10) lead to (3) and (4). (Note that in that case, the partial derivatives of the left side of (10) with respect to u have to vanish by necessity.) By the same token, (3) and (4) together with the transversality conditions (note that beginning point and endpoint of z ($u = z'$) are free) lead to (9) and (10).

However, such a smooth transition process fails when U is not open. Again, it is worthwhile to return for a moment to the simple problem of finding the minimum of a real valued (differentiable) function of a real variable, but this time, in a <u>closed</u> interval: Let $h: [-1,1] \to R$, h differentiable, assume its minimum at $t = t_0 \in [-1,1]$. Unless t_0 is an interior point of which we have no apriori knowledge, (3a) is <u>not</u> a necessary condition anymore. Suppose, we introduce a new independent variable θ by means of the smooth map $\psi : R \xrightarrow{\text{onto}} [-1,1]$ that is given by $\psi(\theta) = \sin\theta$. Then, the function g which is defined by $g(\theta) \equiv h(\sin\theta)$ is defined and differentiable on R and has the same range as h. Now, (3a) and (4a) are valid necessary conditions for g to assume its minimum at $\theta = \theta_0$ and we obtain in turn

(3b) $$g'(\theta_0) \equiv h'(\sin\theta_0)\cos\theta_0 = 0,$$

(4b) $$g(\theta_0) \equiv h(\sin\theta_0) \leq h(\sin\theta) \text{ for all } \theta \in R.$$

The solution of the original problem is then recovered from $t_0 = \sin\theta_0$.

A similar transformation to transform certain (simple) optimal control problems into Lagrange problems has been carried out at occassions by engineers without them ever stopping to realize that they had done something very clever. The author's students, Stephen K. Park, Lawrence M. Hanafy, and Terry A. Straeter*)

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have investigated the potential generalizations of this idea with the goal of establishing the equivalence of optimal control problems with classical Lagrange problems in the sense that with each competing function in the one problem there corresponds at least one competing function in the other problem and vice versa, and, that the integrals to be minimized in the two problems have the same value for corresponding competing functions. Suppose that P_1 is the control problem which we formulated in (7), (8) and that $\psi: R^p \xrightarrow{\text{onto}} U$ is a continuous map from some cartesian space R^p onto the control region $U \in R^r$. Then, if P_2 is the problem of finding a vector function $z': R \to R^p$ so that

(7a) $$x' = \Phi(t,x,\psi(z')),$$

x satisfies the same boundary conditions as in (7), (8), and is such that

(8a) $$J[\psi(z')] = \int_0^T F(t,x(t),\psi(z'(t)))dt$$

yields the smallest possible value, then, since $J[\psi(z')] = J[u]$ for $u = \psi(z')$, the problem P_2 is obviously equivalent to P_1 provided that ψ maps the set of all competing vector functions z' by $u = \psi \circ z'$ onto the set of all competing vector functions u [9,10].

We note in passing that P_2 is a special case of the Lagrange problem (1), (2) with $I[z] \equiv J[\psi(z')]$ and can be dealt with within the framework of the classical theory of the Calculus of Variations. The solutions u of P_1 can then be recovered from the solution z' of P_2 by means of $u(t) = \psi(z'(t))$. Note also that the trajectory x is, in view of the relationship between P_1 and P_2, the same for corresponding functions z' and u.

To be settled is still the question of whether or not there are functions ψ with the required property, to wit: ψ maps the set of all competing functions z' of P_2 onto the set of all competing functions u of P_1.

This problem may be dealt with on various levels. Suppose we consider $z', u \in S$. Our problem may be formulated, without reference to the attending control problems, in the following manner [10]: Given a continuoua map $\psi: R^p \xrightarrow{\text{onto}} U$. Is there, for any given continuous function $u: I \to U$ (where I denotes a closed interval) a continuous (or, maybe, seccionally continuous) function $z': I \to R^p$ such that $\psi(z'(t)) = u(t)$ for all $t \in I$? (See also Figure 2.)

A partial answer to this last question, in form of a (strong) sufficient condition may be found in a paper by E. Floyd [4] where he asserts that ψ has the required property (which topologists call the <u>lifting property</u>) if there is a compact set $Z \subset R^p$ so that $\psi(Z) = U$ and if ψ is open and light on Z, i.e., maps open sets in Z into open sets in U (in the relative topologies of Z and U) and is so that the inverse image in Z of each point in U is totally disconnected. This result is too restrictive for our purpose. First, such maps do not even exist if $m < \dim Z$ and appear to exist for $m > \dim Z$ only for pathological cases, and secondly, even if $\dim Z = p = m$, it eliminates mappings that serve our purpose quite well. (In Figure 3 we depict a mapping $\psi\colon R \xrightarrow{\text{onto}} [-1,1]$ which is a perfectly good mapping for the purpose of transforming the optimal control problem (5), (6) into an equivalent Lagrange problem, but is neither open nor light.)

A more satisfactory answer is available, however, if one allows bounded measurable functions as competing functions and we shall therefore call this, in the time honored tradition, the natural setting for our problem. In this case, a lemma by Filippov [3] provides for a solution to our problem for a large class of optimal control problems. Filippov's lemma, reformulated and specialized to fit our needs, asserts that if U is compact, ψ is continuous, $\psi(R^p) = U$, and if there exists a compact subset $Z \subset R^p$ so that $\psi(Z) = U$, then, for any (bounded) measurable function $u\colon I \to U$ there exists a (bounded) measurable function $z'\colon I \to R^p$ so that $\psi(z'(t)) = u(t)$ for all $t \in I$.

The existence of a function ψ which satisfies the hypotheses of Filippov's lemma depends, of course, on the nature of U and has to be established for various types of compact sets U. (If U is a right parallelepiped, a function ψ with the required property is easy to come by and one may take $p = m$ in this case [9]. If U is a compact, convex polyhedron - a very important type of control region - several such maps which are even smooth, were constructed by Park with $p + 1$ equal to the number of vertices of the polyhedron [10]. This, incidentally, also enabled him to establish the bang-bang principle [11, p. 117] on the basis of (3), (4), and the transversality conditions.)

In all such cases where Filippov's lemma applies, Pontryagin's principle can be derived from the multiplier rule (3), the Weierstrass condition (4) and the transversality conditions [10]. In addition to Pontryagin's principle, one obtains a set of equations, namely

$$[p_0 F_u(t,x(t),\psi(z')) + p\cdot\Phi_u(t,x(t),\psi(z'))]\psi'(z') = 0$$

which one may solve algebraically for z' and thusly obtain from $u = \psi(z')$ the possible candidates for optimal controls - provided that ψ is a smooth map.

It is now tempting to utilize all available knowledge about the calculus of variations, insofar as it pertains to global minima, for the investigation of optimal control problems that are equivalent with Lagrange problems in the above sense. Although some early results concerning the use of Weierstrass' field theory to obtain some sufficient conditions have been somewhat discouraging [6], such investigations are still under way. Also under consideration are questions concerning the applicability of existing algorithms for the numerical solution of optimal control problems with an open control region (or Lagrange problems) to equivalent optimal control problems with a compact control region, and questions about the existence of optimal controls.

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FIG.1. The elephant slide

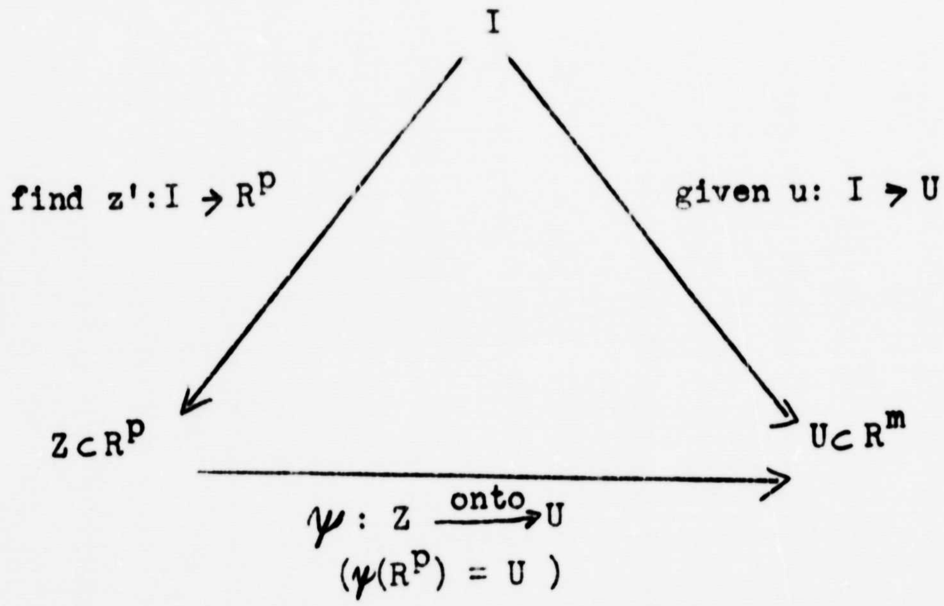

FIG.2. The lifting problem

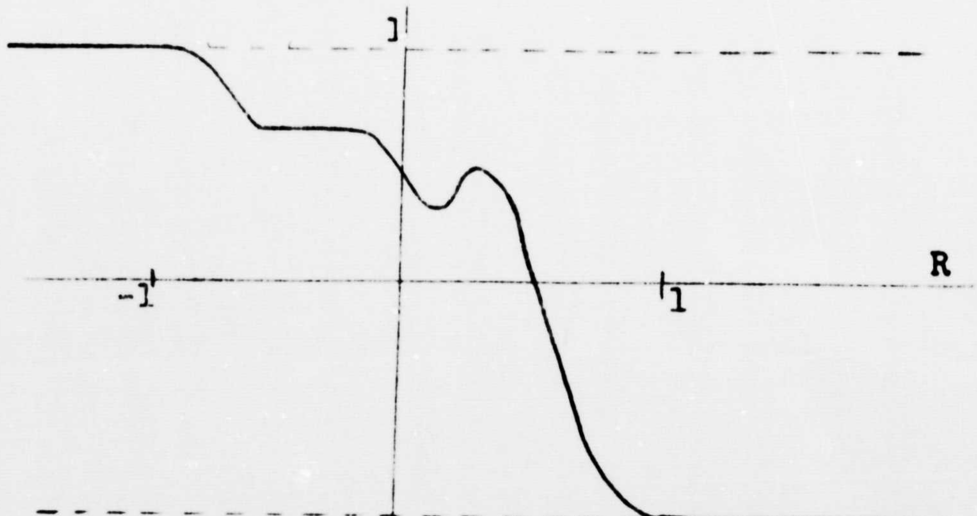

FIG.3. Map that is neither open nor light